An upper bound for Cubicity in terms of Boxicity

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Abstract

An axis-parallel *b*-dimensional box is a Cartesian product $R_1 \times R_2 \times \ldots \times R_b$ where each R_i (for $1 \leq i \leq b$) is a closed interval of the form $[a_i, b_i]$ on the real line. The boxicity of any graph G, $\mathrm{box}(G)$ is the minimum positive integer b such that G can be represented as the intersection graph of axis parallel b-dimensional boxes. A b-dimensional cube is a Cartesian product $R_1 \times R_2 \times \ldots \times R_b$, where each R_i (for $1 \leq i \leq b$) is a closed interval of the form [a_i,a_i+1] on the real line. When the boxes are restricted to be axis-parallel cubes in *b*-dimension, the minimum dimension *b* required to represent the graph is called the cubicity of the graph (denoted by $\mathrm{cub}(G)$). In this paper we prove that $\mathrm{cub}(G) \leq \lceil \log n \rceil \mathrm{box}(G)$ where n is the number of vertices in the graph. We also show that this upper bound is tight.

Keywords: Cubicity, Boxicity, Interval graph, Indifference graph

1 Introduction

Let $\mathcal{F} = \{S_x \subseteq U : x \in V\}$ be a family of subsets of *U*, where *V* is an index set. The intersection graph $\Omega(\mathcal{F})$ of $\mathcal{F}$ has V as vertex set, and two distinct vertices *x* and *y* are adjacent if and only if $S_x \cap S_y \neq \emptyset$. Representations of graphs as the intersection graphs of various geometric objects is a well-studied area in graph theory. A prime example of a graph class defined in this way is the class of interval graphs.

Definition 1. *A graph $I(V,E)$ is an interval graph if and only if there exists a function Π which maps each vertex $u \in V$ to a closed interval of the form $[l(u), r(u)]$ on the real line such that $(u,v) \in E$ if and only if $\Pi(u) \cap \Pi(v) \neq \emptyset$. We will call Π an interval representation of $I(V,E)$.*

Definition 2. *An indifference graph is an interval graph which has an interval representation in which each of the intervals is of the same length. We will call such an interval representation a unit interval representation of the graph.*

The indifference graphs are also known as unit interval graphs. See Chapter 8 of [15] for more information on interval graphs and indifference graphs.

Motivated by theoretical as well as practical considerations, graph theorists have tried to generalize the concept of interval graphs in many ways. In many cases, representation of a graph as the intersection graph of a family of geometric objects, which are generalizations of intervals, is sought. Concepts such as boxicity and interval number are examples.

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In this paper we only consider simple, finite, undirected graphs. $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively.

Definition 3. *For a graph G, $\mathrm{box}(G)$ is the minimum positive integer b such that G can be represented as the intersection graph of axis-parallel b-dimensional boxes. Here a* b*-dimensional box is a Cartesian product $R_1 \times R_2 \times \ldots \times R_b$ where each R_i (for $1 \leq i \leq b$) is defined to be a closed interval of the form $[a_i, b_i]$ on the real line. The boxicity of a complete graph is defined to be 0.*

Definition 4. *The cubicity of a graph G, $\mathrm{cub}(G)$ is the minimum positive integer b such that G can be represented as the intersection graph of axis-parallel b-dimensional cubes. Here a b-dimensional cube is a Cartesian product $R_1 \times R_2 \times \ldots \times R_b$, where each R_i (for $1 \leq i \leq b$) is a closed interval of the form $[a_i, a_i + 1]$ on the real line. The cubicity of a complete graph is defined to be 0.*

The following observation is easy to make. A 1-dimensional box is a closed interval on the real line and thus graphs of boxicity 1 are exactly the interval graphs. Similarly, the graphs with cubicity 1 are the indifference graphs.

Lemma 1 (Roberts[19]). *Given a graph G,the minumum positive integer b such that there exist interval graphs $G_1, G_2, \ldots G_b$ with $V(G) = V(G_i)$ for $1 \leq i \leq b$ and satisfying $E(G) = E(G_1) \cap E(G_2) \cap \ldots E(G_b)$ is equal to $\mathrm{box}(G)$.*

Lemma 2 (Roberts[19]). *Given a graph G,the minumum positive integer b such that there exist indifference graphs $G_1, G_2, \ldots G_b$ with $V(G) = V(G_i)$ for $1 \leq i \leq b$ and satisfying $E(G) = E(G_1) \cap E(G_2) \cap \ldots E(G_b)$ is equal to $\mathrm{cub}(G)$.*

The concepts of cubicity and boxicity were introduced by F.S. Roberts [19]. They find applications in niche overlap in ecology and in solving problems of fleet maintanence in operations research. (See [11].) It was shown by Cozzens [10] that computing the boxicity of a graph is an NP-hard problem. Later, this was improved by Yannakakis[23] , and finally by Kratochvil[17] who showed that deciding whether the boxicity of a graph is at most 2 itself is an NP-complete problem. The complexity of finding the maximum independent set in bounded boxicity graphs was considered by [16, 14]. Some NP-hard problems are known to be either polynomial time solvable or have much better approximation ratio on low boxicity graphs. For example, the max-clique problem is polynomial time solvable on bounded boxicity graphs and the maximum independent set problem has $\log n$ approximation ratio for graphs with boxicity 2 [1, 3].

There have been many attempts to find the cubicity and boxicity of graphs with special structures. In his pioneering work, F.S. Roberts[19] proved that the boxicity of a complete *k*-partite graph (where each part has at least 2 vertices) is *k*. He also proved that the cubicity of any graph can not be greater than $\lfloor 2n/3 \rfloor$ and the boxicity cannot be greater than $\lfloor n/2 \rfloor$. Scheinerman[20] showed that the boxicity of outer planar graphs is at most 2. Thomassen[21] proved that the boxicity of planar graphs is bounded above by 3. The boxicity of split graphs is investigated by Cozzens and Roberts[11]. Chandran and Sivadasan[6] proved that the cubicity of the d-dimensional hypercube H_d is $\theta(\frac{d}{\log d})$. They also proved that for any graph G, $\mathrm{box}(G) \leq tw(G) + 2$ where $tw(G)$ is the treewidth of G [7]. This in turn throws light on the boxicity of various other graph classes. Roberts and Cozzens proposed a theory of dimensional properties, attempting to generalize the concepts of cubicity and boxicity [12]. These concepts were further developed by Kratochvil and Tuza [18].

Researchers have also tried to generalize or extend the concept of boxicity in various ways. The poset boxicity [22] , the rectangular number [8] , grid dimension [2], circular dimension[13] and the boxicity of digraphs[9] are some examples.

2 Our Results

It is easy to see that for any graph G, $\mathrm{box}(G) \leq cubi(G)$. In this paper we prove the following theorem:

Theorem 1. *For a graph G on n vertices, $\mathrm{cub}(G) \leq \lceil \log n \rceil \mathrm{box}(G)$. Moreover, this upper bound is tight.*

2.1 Consequences of our result

The upper bound that we developed should be useful in many cases where a bound for one of the two quantities (boxicity and cubicity) is already known. Combining our theorem with previously known upper bounds for boxicity, we get various upper bounds for cubicity, which we list in the following table. Here n denotes the number of vertices in the graph, $tw = treewidth(G)$ is the treewidth of G, $\Delta = \Delta(G)$ is the maximum degree and $\omega = \omega(G)$ is the clique number, i.e. the number of vertices in the biggest clique in G. Each of the references given corresponds to the paper in which the corresponding upper bound for boxicity was proved.

Graph Class	Upper bound for $\mathrm{box}(G)$	Upper bound for $\mathrm{cub}(G)$
Chordal Graphs[7]	$\omega + 1$ $\Delta + 2$	$(\omega + 1) \log n$ $(\Delta + 2) \log n$
Circular Arc Graphs[7]	$2\omega + 1$ $2\Delta + 3$	$(2\omega + 1) \log n$ $(2\Delta + 3) \log n$
AT-Free Graphs[7]	3Δ	$(3\Delta) \log n$
Co-comparability graphs[7]	$(2\Delta + 1)$	$(2\Delta + 1) \log n$
Permutation Graphs[7]	$(2\Delta + 1)$	$(2\Delta + 1) \log n$
Planar Graphs[17]	3	$3 \log n$
Series Parallel Graphs[4]	3	$3 \log n$
Outer Planar Graphs[20]	2	$2 \log n$
Any Graph[7]	$tw + 2$	$(tw + 2) \log n$
Any Graph[5]	$(\Delta + 2) \log n$	$(\Delta + 2) \log^2 n$

2.1.1 Algorithmic Consequences

Our proof provides an $O(n^2 \log n)$ algorithm to represent any interval graph G (on n vertices) into a $\log n$-space as the intersection graph of n axis parallel $\log n$-dimensional cubes, when the interval representation of G is given. Also follows from this, a polynomial time algorithm to translate any given box representation of a graph in a b-dimensional space to a cube representation in $b \log n$-dimensional space.

3 Proof of our Theorem

Lemma 3 (Roberts[19]). *Let G be a graph and let $G_1, G_2, \cdots, G_j$ be graphs such that* (1) $V(G) = V(G_p)$ *for* $1 \leq p \leq j$ **and** (2) $E(G) = E(G_1) \cap E(G_2) \cap \ldots E(G_j)$. *Then* $\mathrm{cub}(G) \leq \mathrm{cub}(G_1) + \mathrm{cub}(G_2) + \ldots + \mathrm{cub}(G_j)$.

Lemma 4. *Let $r(n)$ denote the largest real number such that there exists a non-complete graph G (i.e. a graph G such that $\mathrm{box}(G) > 0$) on n vertices such that $\mathrm{cub}(G) = r(n)\mathrm{box}(G)$. Then, there exists an interval graph G' on n vertices such that $\mathrm{cub}(G') = r(n)$.*

Proof. Let G be a graph on n vertices such that $\mathrm{box}(G) = b$ and $\mathrm{cub}(G) = b \cdot r(n)$. Then by Lemma 1, there exists interval graphs $G_1, G_2, \cdots, G_b$ such that $V(G_i) = V(G)$ for $1 \le i \le b$ and $E(G) = E(G_1) \cap E(G_2) \cap \ldots E(G_b)$. By Lemma 3, $r(n) \cdot b = \mathrm{cub}(G) \le \sum_{i=1}^{b} \mathrm{cub}(G_i)$. It follows that there exists at least one i, ($1 \le i \le b$) such that $\mathrm{cub}(G_i) \ge r(n)$. Recallin that G_i is a (non-complete) interval graph and thus $\mathrm{box}(G_i) = 1$ we have $\mathrm{cub}(G_i) \ge r(n) \cdot \mathrm{box}(G_i)$. From the definition of $r(n)$, it follows that $\mathrm{cub}(G_i) = r(n) \cdot \mathrm{box}(G_i) = r(n)$, as required. □

Lemma 5. *For every interval graph G on n vertices, there exists an ordering $f : V(G) \to \{1, 2, \cdots, n\}$ of its vertices such that if $u, v, w \in V(G)$ satisfy $f(u) < f(w) < f(v)$ and $(u, v) \in E(G)$ then $(u, w) \in E(G)$, also.*

Proof. Consider an interval representation of G and order the vertices in the non-decreasing order of the left end-points of the intervals. It is easy to verify that this order satisfies the required property. □

Proof of Theorem 1

By Lemma 4, it is enough to show that for any interval graph G on n vertices, $\mathrm{cub}(G) \le \lceil \log n \rceil$. Let us first assume that $n = 2^k$ for a positive integer k. (We will take care of the remaining case in the end.) Then by Lemma 2, we only have to show that there exists k indifference graphs $I_1, I_2, \cdots, I_k$ such that $V(I_i) = V(G)$ for $1 \le i \le k$ and $E(G) = \bigcap_{i=1}^{k} E(I_i)$. Let f be an ordering of V as described in Lemma 5. First we define $k+1$ different partitions $\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_{k+1}$ of V as follows:

$$\mathcal{P}_i = \{S_1^i, S_2^i, \ldots S_{m_i}^i\}, \textbf{ where } S_j^i = \{v \in V : (j-1)2^{i-1} + 1 \le f(v) \le j 2^{i-1}\}$$

The reader can easily verify that for each i, $1 \le i \le k+1$, $\mathcal{P}_i$ defines a valid partition of V i.e., $\bigcup_j S_j^i = V$ and $S_a^i \cap S_b^i = \emptyset$ for $a \ne b$. Moreover for partition $\mathcal{P}_i$ all blocks have same cardinality, i.e. $|S_j^i| = 2^{i-1}$. Moreover $m_i = 2^{k-i+1}$. For $i \le k$, m_i is an clearly an even number. The partition $\mathcal{P}_{k+1}$ contains only one block, namely $S_1^{k+1} = V$.
For $1 \le i \le k$, we construct the indifference graph I_i based on the partition $\mathcal{P}_i$. Let

$$A_i = S_1^i \cup S_3^i \cup \ldots \cup S_{m_i-1}^i \text{ and } B_i = S_2^i \cup S_4^i \cup \ldots \cup S_{m_i}^i$$

Clearly (A_i, B_i) is a partition of V. Now we define the indifference graph I_i by defining its unit interval representation Π_i as follows:

For $v \in B_i$: $\Pi_i(v) = [n + f(v), 2n + f(v)]$.
For $v \in A_i$, if $N(v) \cap B_i = \emptyset$: $\Pi_i(v) = [0, n]$.
For $v \in A_i$, if $N(v) \cap B_i \ne \emptyset$: (Let $t = max_{x \in N(v) \cap B_i} f(x)$.) $\Pi_i(v) = [t, n + t]$

Claim 1. $E(I_i) \supseteq E(G)$ *for* $1 \le i \le k$

Let $(u, v) \in E(G)$. We only have to consider the following three cases.
Case 1: $u \in A_i$ and $v \in A_i$. Then $\Pi_i(u) \cap \Pi_i(v) \ne \emptyset$ since the point corresponding to n on the real line is a member of both $\Pi_i(u)$ and $\Pi_i(v)$.
Case 2:$u \in B_i$ and $v \in B_i$. Here also $\Pi_i(u) \cap \Pi_i(v) \ne \emptyset$ since the point corresponding to $2n$ on the real line is a member of both $\Pi_i(u)$ and $\Pi_i(v)$.
Case 3: $u \in A_i$ and $v \in B_i$. In this case, let $z = \max(f(x) : x \in N(u) \cap B_i)$. Now, $f(v) \le z$, since $v \in N(u) \cap B_i$. Now recall that $\Pi_i(v) = [n + f(v), 2n + f(v)]$ and $\Pi_i(u) = [z, n + z]$. Clearly, the point corresponding to $n + z$ on the real line belongs to both $\Pi_i(u)$ and $\Pi_i(v)$, and thus $\Pi_i(u) \cap \Pi_i(v) \ne \emptyset$. ■

Claim 2. *If $(u, v) \notin E(G)$ then there exists an i, $1 \leq i \leq k$ such that $(u, v) \notin E(I_i)$.*

Let t be the largest integer such that for $1 \leq i \leq t$, u and v belong to different blocks of the partion $\mathcal{P}_i$, i.e. if $1 \leq i \leq t$ and $u \in S_a^i$ and $v \in S_b^i$, then $a \neq b$. Clearly such a t exists and in fact $t \leq k$, since $\mathcal{P}_{k+1}$ contains only one block. Without loss of generality, assume that $f(u) < f(v)$. We claim that if $u \in S_a^t$ and $v \in S_b^t$ then $b = a + 1$, where a is an odd number. To see this notice that by the definition of t, u and v belong to the same block in P_{t+1} and if $u, v \in S_c^{t+1}$ then clearly $u \in S_a^t$ and $v \in S_b^t$, where $a = 2(c-1) + 1$ and $b = 2(c-1) + 2$.

Now we will show that $(u, v) \notin E(I_t)$. To see this, first observe that $u \in A_t$ and $v \in B_t$ since $u \in S_a^t$ where a is an odd number and $v \in S_{a+1}^t$ where $a + 1$ is an even number. If $N(u) \cap B_t = \emptyset$, clearly $(u, v) \notin E(I_t)$, since in that case $\Pi_t(u) = [0, n]$ and $\Pi_t(v) = [n + f(v), 2n + f(v)]$ and these two intervals do not intersection. So, we can assume that $N(u) \cap B_t \neq \emptyset$. Now, let $w \in B_t$ be such that $f(w) = \max(f(x) : x \in N(u) \cap B_t)$. We claim that $f(w) < f(v)$. Suppose not. Then clearly $f(u) < f(v) < f(w)$. Now by Lemma 5, $(u, v) \in E(G)$, since $(u, w) \in E(G)$, contradicting the assumption that $(u, v) \notin E(G)$. Now, recall that $\Pi_t(u) = [f(w), n + f(w)]$ and $\Pi_t(v) = [n + f(v), 2n + f(v)]$. Since $f(w) < f(v)$ we have $\Pi_t(u) \cap \Pi_t(v) = \emptyset$ and thus $(u, v) \notin E(I_t)$. ■

From Claim 1 and Claim 2 we have, $E(G) = E(I_1) \cap E(I_2) \cap \ldots E(I_k)$ as required. So by Lemma 2, $\mathrm{cub}(G) \leq k = \log n$. If $2^{k-1} < |V| < 2^k$, then add $2^k - |V|$ isolated vertices to the graph. Note that this will not change the cubicity or boxicity of the graph. Moreover $\lceil \log n \rceil = k$, and the result follows.

Finally the tightness of our result can be verified by considering the star graph on n vertices, $S(n)$. (Note: The star graph $S(n)$ is the complete bipartite graph $K_{1,n-1}$, with a single on one side and the remaining $n - 1$ vertices on the other side.) Its boxicity equals 1, since it is an interval graph. It is also known that [19] $\mathrm{cub}(S(n)) = \lceil \log(n-1) \rceil$. Note that when $n \neq 2^k + 1$, we have $\lceil \log(n-1) \rceil = \lceil \log n \rceil$ and thus our upper bound is tight. □

Remark 1. *The k indifference graphs that we constructed all have a diameter less than or equal to 2. Thus it follows from our proof that the edge set of any inteval graph can be represented as the intersection of the edge sets of at most $\lceil \log n \rceil$ indifference graphs of diameter at most 2.*

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